

VARIATIONAL EQUATIONS ON MIXED RIEMANNIAN-LORENTZIAN METRICS

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ABSTRACT. A class of elliptic-hyperbolic equations is placed in the context of a geometric variational theory, in which the change of type is viewed as a change in the character of an underlying metric. A fundamental example of a metric which changes in this way is the extended projective disc, which is Riemannian at ordinary points, Lorentzian at ideal points, and singular on the absolute. Harmonic fields on such a metric can be interpreted as the hodograph image of extremal surfaces in Minkowski 3-space. This suggests an approach to generalized Plateau problems in 3-dimensional space-time via Hodge theory on the extended projective disc. Analogous variational problems arise on Riemannian-Lorentzian flow metrics in fiber bundles (twisted nonlinear Hodge equations), and on certain Riemannian-Lorentzian manifolds which occur in relativity and quantum cosmology. The examples surveyed come with natural gauge theories and Hodge dualities. This paper is mainly a review, but some technical extensions are proven. *MSC2000*: 35M10, 53A10, 83C80

Key words: signature change, projective disc, Minkowski 3-space, equations of mixed type, nonlinear Hodge equations

1. INTRODUCTION: THE PROJECTIVE DISC

In a small circle of paper, you shall see as it were an epitome of the whole world.

Giambattista Della Porta, 1589, on the *camera obscura*

Analysis on Beltrami's projective disc model for hyperbolic space is in one sense very old mathematics. Beltrami introduced the projective disc in 1868 as one of the earliest Euclidean models for non-Euclidean space [9]; see also [10], [113]. But it also arises in the context of some new mathematics related to variational problems in Minkowski space and Hodge theory on pseudo-Riemannian manifolds. In practice Beltrami's construction amounts to equipping the unit disc centered at the origin of coordinates in \mathbb{R}^2 with the distance function

$$(1) \quad ds^2 = \frac{(1 - y^2) dx^2 + 2xy dx dy + (1 - x^2) dy^2}{(1 - x^2 - y^2)^2}.$$

Integrating ds along geodesic lines in polar coordinates, we find that the distance from any point in the interior of the unit disc to the boundary of the disc is infinite, so the unit circle becomes the *absolute*: the curve at projective infinity; *c.f.* [49], Sec. 9.1.

It is natural to ask how to interpret the so-called *ideal* points in the complement of the unit disc in \mathbb{R}^2 . It has been known for a long time that such points are not merely allowed by the projective disc model in a formal sense, but are actually useful in classical geometric constructions. For example, tangent lines to the unit disc can be used to characterize orthogonal lines within the disc, and certain families

of translated lines inside the disc attain their simplest representation as a rotation about an ideal point. Although these classical geometric operations on ideal points are well known [112], [64], geometric analysis on domains which include ideal points is still very incompletely understood.

In this review we use Beltrami's model as a point of reference from which to survey aspects of geometric variational theory on *mixed Riemannian-Lorentzian* domains, on which the signature of the metric changes sign along a smooth hypersurface. (The metric underlying (1) changes from Riemannian to Lorentzian on the Euclidean unit circle.) In Sec. 2 we consider variational problems which reduce to the existence of harmonic fields on the extended projected disc – that is, solutions to the *Hodge equations*

$$(2) \quad d\alpha = \delta\alpha = 0$$

on a domain of \mathbb{R}^2 which includes the closed unit disc as a proper subset and is equipped with the Beltrami metric (1). Here d is the exterior derivative, with formal adjoint δ , acting on a 1-form α . The crucial technical problem for the equations of Sec. 2 is the existence of solutions to boundary-value problems; this question is addressed in Sec. 3. Sections 4 and 5 are concerned with the related topics of duality and gauge invariance. These topics motivate the study of gauge-invariant, potentially elliptic-hyperbolic systems in fiber bundles. In Sec. 6 we briefly review similar systems that arise in relativity and cosmology, and the relation of those systems to other elliptic-hyperbolic equations on manifolds which have been studied by mathematicians.

On the extended projective disc the Hodge equations are no longer uniformly elliptic, but are elliptic on ordinary points inside the unit disc, hyperbolic on ideal points in the \mathbb{R}^2 -complement of the disc, and parabolic on the unit circle, which is a singularity of the manifold. One might wonder why anyone would want to study such a peculiar system. There are at least two motivations for doing so:

- i)* to learn what the geometry of the Beltrami disc reveals about elliptic-hyperbolic partial differential equations, and
- ii)* to learn what elliptic-hyperbolic partial differential equations on the Beltrami disc reveal about the geometry of space-time.

Regarding the first motivation, there is no canonical way to decide what constitutes a natural boundary-value problem for an equation that changes from elliptic to hyperbolic type on a smooth curve. Historically, physical analogies have been the main tool, chiefly analogies to the physics of compressible flow [12]. However, it is also possible to approach the problem using a geometric analogy, which we will briefly describe.

The highest-order terms of any linear second-order partial differential equation on a domain $\Omega \subset \mathbb{R}^2$ can be written in the form

$$(3) \quad Lu = \alpha(x, y) u_{xx} + 2\beta(x, y) u_{xy} + \gamma(x, y) u_{yy},$$

where (x, y) are coordinates on Ω ; α , β , and γ are given functions; $u = u(x, y)$. Traditionally, the question of whether the equation is elliptic, hyperbolic, or parabolic has been identified with the question of whether the discriminant

$$(4) \quad \Delta(x, y) = \alpha\gamma - \beta^2$$

is respectively positive, negative, or zero. If the discriminant is positive on part of Ω and negative elsewhere on Ω , then the equation associated with the operator

L is said to be of *mixed elliptic-hyperbolic type*. The curve on which the equation changes type is called the *parabolic line* or, borrowing the terminology of fluid dynamics, the *sonic curve*. A simple example of an elliptic-hyperbolic equation is the *Lavrent'ev-Bitsadze equation*

$$\operatorname{sgn}(y)u_{xx} + u_{yy} = 0,$$

for which the parabolic line is the x -axis.

An alternative approach is to replace the class of differential operators L on a single domain of \mathbb{R}^2 with the single *Laplace-Beltrami operator*,

$$\mathcal{L}_g u = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{|g|} \frac{\partial u}{\partial x^j} \right),$$

on a class of domains for which g_{ij} represents the local metric tensor.

For example, the Lavrent'ev-Bitsadze equation can be associated to the Laplace-Beltrami operator on a metric which is Euclidean above the x -axis and Minkowskian below the x -axis. Elliptic operators will be associated to Riemannian metrics; wave operators will be associated to Lorentzian metrics. In this classification, the type of a linear second-order equation is not a function of the associated linear operator at all; that operator is always the Laplace-Beltrami operator. Rather, the type of the equation is a feature of the metric tensor on the underlying surface. Any change in the signature which results in a change in sign of the determinant g will change the Laplace-Beltrami operator on the metric from elliptic to hyperbolic type. The Laplace-Beltrami operator on surface metrics for which such a change occurs along a smooth curve will correspond to planar elliptic-hyperbolic operators in local coordinates. However, any curve on which the change of type occurs will necessarily represent a singularity of the metric tensor, as the determinant g will vanish along that curve.

According to this point of view, in order to decide which boundary-value problems are natural for a second-order linear elliptic-hyperbolic equation on \mathbb{R}^2 , one should study the geometry of the underlying pseudo-Riemannian metric. For example, characteristic lines for the Hodge equations on extended \mathbb{P}^2 can be interpreted as polar lines for a chord of the projective disc, and this suggests a natural class of boundary-value problems [95].

However, in this review we focus on the second motivation for studying harmonic fields on the extended projective disc. This is the opportunity that they afford to revisit, from a different point of view, work done by Chao-Hao Gu in the 1980's on the existence of extremal surfaces in Minkowski 3-space $\mathbb{M}^{2,1}$ [41]-[45], and similar variational problems which arise in the context of fiber bundles.

2. THE GEOMETRIC VARIATIONAL PROBLEM

The area functional for a smooth surface Σ in $\mathbb{M}^{2,1}$ having graph $z = f(x, y)$ is given by

$$A = \int \int_{\Sigma} \sqrt{|1 - f_x^2 - f_y^2|} \, dx dy.$$

The surface Σ is time-like when $f_x^2 + f_y^2$ exceeds unity and space-like when $f_x^2 + f_y^2$ is exceeded by unity. Introducing Lagrange's notation $p = f_x$, $q = f_y$, the boundary between the space-like and time-like surfaces is the unit circle centered at the origin of coordinates in the pq -plane.

A necessary condition for Σ to be extremal on $\mathbb{M}^{2,1}$ is that its graph $f(x, y)$ satisfy the minimal surface equation in the form [42]

$$(5) \quad (1 - p^2) q_y + 2pq p_y + (1 - q^2) p_x = 0.$$

This is a quasilinear partial differential equation which is elliptic for space-like surfaces and hyperbolic for time-like surfaces. We can linearize this equation by the method of Legendre ([23], Sec. I.6.1), applying the transformation $z = px + qy - \varphi(p, q)$, $x = \varphi_p$, $y = \varphi_q$. We obtain the linear equation

$$(1 - p^2) \varphi_{pp} - 2pq \varphi_{pq} + (1 - q^2) \varphi_{qq} = 0.$$

Adopting homogeneous coordinates (u, v, w) for $w \neq 0$, we eventually obtain the equation [43]

$$(6) \quad [(1 - p^2) \psi_p]_p - 2pq \psi_{pq} + [(1 - q^2) \psi_q]_q = 0,$$

where $p = -u/w$ and $q = -v/w$. Equation (6) can be interpreted as the Laplace-Beltrami equation on the extended projective disc \mathbb{P}^2 ; *c.f.* [95] and references therein.

In models for more general classes of surfaces (*e.g.*, Examples 2-4 from Sec. 2 of [109]), the gradient of f may be replaced by the d -closed 1-form $\omega = \omega_1 dx + \omega_2 dy$. In particular, consider a space-like surface Σ embedded in $\mathbb{M}^{2,1}$. The variational equations of the area functional can be written in the form

$$(7) \quad [(1 - \omega_2^2) \omega_1]_x + \omega_1 \omega_2 (\omega_{1y} + \omega_{2x}) + [(1 - \omega_1^2) \omega_2]_y = 0,$$

$$(8) \quad \omega_{2x} - \omega_{1y} = 0.$$

These are the equations for an extremal surface which is space-like in some regions of $\mathbb{M}^{2,1}$ and time-like in others. However, the surface that it describes is singular on the circle $\omega_1^2 + \omega_2^2 = 1$.

If the Gaussian curvature of the surface is nonvanishing, then

$$\omega_{1x} \omega_{2y} - (\omega_{1y})^2 = \omega_{1x} \omega_{2y} - \omega_{1y} \omega_{2x} \neq 0$$

and the nonlinear system can be linearized by a Legendre transformation to obtain

$$(9) \quad [(1 - x^2) u_1]_x - (xy u_1)_y - (xy u_2)_x + [(1 - y^2) u_2]_y = 0,$$

$$(10) \quad u_{1y} - u_{2x} = 0.$$

This system has a geometric interpretation as the Hodge equations on the extended projective disc.

From this point of view, solving the Hodge equations locally on extended \mathbb{P}^2 is a way to approach generalized Plateau problems in Minkowski 3-space. For example, suppose one were to study the existence question for a class of area-extremizing surfaces in $\mathbb{M}^{2,1}$. The first step in such a program might be an existence theorem for a class of *Dirichlet problems* for eqs. (9), (10): The hypothesis of a sufficiently smooth vector-valued function h prescribed on all or part of a boundary curve in extended \mathbb{P}^2 would be shown to imply the existence of a 1-form satisfying the Hodge equations in the interior and equalling h on the boundary.

Unfortunately, a good Hodge theory on extended \mathbb{P}^2 would not in itself provide a good variational theory even for stationary hypersurfaces in $\mathbb{M}^{2,1}$. One reason is that the Legendre transformation may itself introduce singularities. It is known

that, in the elliptic region, such singularities can only occur on at most a point set if the image is the system (9), (10) [93]. However, higher-order singularities in the parabolic and hyperbolic regions of the equations are possible and, based on analogies to the equations of gas dynamics [22], expected. Moreover, although the Legendre transformation makes the equation simpler, it makes boundary conditions more complicated, so the interpretation of boundary-value problems for harmonic 1-forms on extended \mathbb{P}^2 in terms of extremal hypersurfaces in $\mathbb{M}^{2,1}$ is not generally straightforward.

Nevertheless, the Dirichlet problem for a quasilinear elliptic-hyperbolic system, having a line singularity on the parabolic curve, is sufficiently formidable that the approach via linearization remains the one with the most apparent promise. (For a possible alternative, see [118] and references therein; for an alternative in the particular case studied by Gu, see [65].)

3. BOUNDARY-VALUE PROBLEMS

Both of the two motivations for studying the Hodge equations on extended \mathbb{P}^2 begin with the existence question for boundary-value problems: under what conditions on the boundary can we expect solutions in the interior? The obstruction to answering this question is the problematic nature of boundary-value problems for elliptic-hyperbolic equations.

The systematic study of partial differential equations which change from elliptic to hyperbolic type on a smooth hypersurface began in 1923 with the famous memoir of Tricomi [117] concerning the equation

$$(11) \quad yu_{xx} + u_{yy} = 0.$$

Tricomi's work was extended in the early 1930s by Cinquini-Cibrario [21] to include the equation

$$(12) \quad xu_{xx} + u_{yy} = 0.$$

The study of these equations led to the establishment of two classes of normal forms, to which any linear elliptic-hyperbolic equation of second order can be reduced near a point (*c.f.* [13]), and some appropriate boundary-value problems for such forms. In particular, the *Tricomi problem* for (11) or (12) prescribes the solution on the part of the boundary consisting of characteristic curves; see, *e.g.*, Sec. 12.1 of [34] for a discussion. Tricomi problems for the scalar case of the system (9), (10) were solved by Hua and his students in the 1980s [53], [55].

The boundary-value problems formulated by Tricomi and Cinquini-Cibrario were *open* in the sense that the boundary conditions are prescribed on a proper subset of the boundary. *Closed* problems, in which data are prescribed on the entire boundary, have only recently been studied in a systematic way [70] see also [103]. Closed problems were formulated as early as 1929 by Bateman [8], but in the informal style typical of British applied mathematics of that period. An isolated result, the existence of weak solutions to the closed Dirichlet problem for the scalar Tricomi equation, was proven by Morawetz in 1967 [81]; see also [104]. Some special cases are treated in [41], [71], [98], and [115].

The closed boundary-value problems in [70] are formulated for a class of equations having the form

$$(13) \quad K(y)u_{xx} + u_{yy} = 0,$$

where the *type-change function* $K(y)$ is a continuously differentiable function satisfying certain technical properties, the most important of which are $K(0) = 0$ and $yK(y) > 0$ for $y \neq 0$. In the simplest special case, $K(y) = y$, (13) reduces to the Tricomi equation (11). For this reason, equations of the form (13) are said to be of *Tricomi type*. Similarly, eq. (12) is the simplest example of an equation of *Keldysh type* – that is, an equation of the form [57]

$$(14) \quad K(x)u_{xx} + u_{yy} = 0,$$

where $K(x)$ is generally taken to be a continuously differentiable function such that $K(0) = 0$ and $xK(x) > 0$ for $x \neq 0$. Equations of this kind, with various lower-order terms, have arisen in the study of transonic gas dynamics and, recently, in optics; see, *e.g.*, Sec. 3 of [16] and Sec. 4 of [71], respectively. If the components of the vector (u_1, u_2) are sufficiently differentiable, then the system (9), (10) in polar coordinates can be represented as a single equation of Keldysh type. Sufficient conditions for the existence of unique solutions to closed boundary-value problems for equations having the general form (14) are virtually unknown.

The known results on the existence of solutions to boundary-value problems for (9), (10) can be summarized as follows:

Let Ω be the domain formed by the polar lines of a chord of the unit disc in extended \mathbb{P}^2 . Then there exists a weak solution on Ω with data prescribed on the non-characteristic part of the boundary. In fact, we can deform the chord in such a way that there is also a non-characteristic, explicitly hyperbolic boundary on which data are prescribed, providing a mild monotonicity condition is met. Solutions lie in a weighted function space. The weak solution is strong (in the sense of Sec. 3.2, below) if we round off the sharp points on the boundary and perturb the lower-order terms. These assertions are stated precisely and proven in Sec. II of [95]. Although they are formulated in [95] in the context of projective geometry, a (different) boundary-value problem for an analogous domain in the scalar case has been formulated in the context of Minkowski geometry; compare Fig. 1 of [44] with Sec. 4, Fig. 2 of [53].

However, we show in the following section that the closed Dirichlet problem for twice-continuously differentiable solutions of (9), (10) is over-determined on the hyperbolic boundary. Note the odd terminology: Hyperbolic *points* are ordinary points lying inside the unit disc, and thus lie in the elliptic region of the eqs. (9), (10). The hyperbolic *boundary* is the portion of the domain boundary on which the discriminant (4) of the equation is negative; so the hyperbolic boundary for (9), (10) lies outside the unit disc and consists of ideal points rather than hyperbolic points.

3.1. The nonexistence of classical solutions. Expressed in polar coordinates, eq. (6) assumes the form

$$(15) \quad (1 - r^2) \phi_{rr} + \frac{1}{r^2} \phi_{\theta\theta} + \left(\frac{1}{r} - 2r \right) \phi_r = 0,$$

for $\phi = \phi(r, \theta)$, provided $r \neq 0$. We define eq. (15) over a sector

$$\Omega_s = \{(r, \theta) \mid 0 < r_0 \leq r \leq r_1, \theta_1 \leq \theta \leq \theta_2\},$$

where $\theta_2 - \theta_1 < \pi$ and the interior of Ω_s includes a segment of the line $r = 1$. For convenience we will take θ_1 to be negative and θ_2 to be positive.

Writing eq. (15) in the equivalent form

$$(16) \quad r^2(1 - r^2)\phi_{rr} + \phi_{\theta\theta} + r(1 - 2r^2)\phi_r = 0,$$

we show that the Dirichlet problem for this equation on a typical domain has a unique solution if data are given on only the non-characteristic portion of the boundary. This will imply that the Dirichlet problem is over-determined if data are prescribed on the entire boundary.

Define the set $\Omega^+ \subset \Omega_s$, where

$$\Omega^+ = \{(r, \theta) \in \mathbb{R}^2 \mid r_0 < \varepsilon \leq r < 1, -\theta_0 \leq \theta \leq \theta_0\}.$$

We will choose the domain Ω of eq. (16) to be the region enclosed by the annular sector Ω^+ and the intersecting lines tangent to the points $(r, \pm\theta) = (1, \pm\theta_0)$.

Precisely, let $(r, \pm\theta) = (1, \pm\theta_0)$ be the two points of intersection of the line $x = x_0$, $x_0 \in (\varepsilon, 1)$, with the unit circle centered at the origin of coordinates in \mathbb{R}^2 . Let Ω_0 be the triangular region bounded by the vertical chord γ_0 given in Cartesian coordinates by

$$\gamma_0 = \{(x, y) \in \mathbb{R}^2 \mid x = x_0, y^2 \leq 1 - x_0^2\}$$

and the two polar lines of γ_0 . (Recall that the *polar* lines of a chord are the tangent lines to the unit circle at its two points of intersection with the chord.) Then $\Omega = \Omega^+ \cup \Omega_0$.

In the following we denote by Ω^- the subdomain of Ω_0 consisting of ideal points, and by ν the arc of the unit circle lying between the points $(r, \pm\theta) = (1, \pm\theta_0)$. Because ε is a fixed number greater zero, the mapping from the region Ω in the polar $r\theta$ -plane (the Cartesian xy -plane) to its image in the Cartesian $r\theta$ -plane is well-defined, and we shall call the image Ω as well.

Theorem 1. *Considering (16) as an equation on the subdomain Ω of the Cartesian $r\theta$ -plane, any solution $\phi \in C^2(\Omega)$ of (16) taking values $f(r, \theta) \in C^2(\partial\Omega)$ on the boundary segment $\partial\Omega \setminus \Omega^-$ is unique.*

Proof. The method of proof has been applied to other elliptic-hyperbolic equations [72], [79], [82]. Suppose that there are two solutions satisfying the boundary conditions. Subtraction yields a solution, which we also denote by ϕ , satisfying homogeneous boundary conditions. Define the functional

$$I = \int^{(r, \theta)} \psi_1 d\theta + \psi_2 dr,$$

where

$$(17) \quad \psi_1 = r^2(1 - r^2)\phi_r^2 - \phi_\theta^2$$

and

$$(18) \quad \psi_2 = -2\phi_r\phi_\theta,$$

on the Cartesian $r\theta$ -plane. Because

$$\psi_{2\theta} - \psi_{1r} =$$

$$(19) \quad -2\phi_r [r^2 (1 - r^2) \phi_{rr} + \phi_{\theta\theta} + r (1 - 2r^2) \phi_r] = 0,$$

there is a function $\chi(r, \theta)$ such that $\chi_\theta = \psi_1$ and $\chi_r = \psi_2$.

The first step is to show that ϕ vanishes identically on ν . Because ϕ is the difference of two solutions having identical boundary values, $\phi(1, \pm\theta_0) = 0$. Also, ϕ_r is zero along the lines $\theta = \pm\theta_0$. Then $\chi_r = \psi_2 = -2\phi_r\phi_\theta = 0$ on these lines. Integrating, $\chi(r, -\theta_0) = c_1$ and $\chi(r, \theta_0) = c_2$, where c_1 and c_2 are constants. On the arc ν ,

$$(20) \quad \chi_\theta = \psi_1 = -\phi_\theta^2 \leq 0$$

so $c_1 \geq c_2$. On the intersection of the line $r = \varepsilon$ with $\partial\Omega$,

$$\chi_\theta = \psi_1 = \varepsilon (1 - \varepsilon^2) \phi_r^2 \geq 0,$$

as ϕ_θ is zero there by the homogeneous boundary conditions, implying that $c_2 \geq c_1$. We conclude that $c_1 = c_2$. Now (20) implies that $\chi_\theta \equiv 0$ on ν . Integrating the differential equation $\chi_\theta = -\phi_\theta^2 = 0$ and using the boundary conditions $\phi(1, \pm\theta_0) = 0$, we conclude that ϕ vanishes identically on the parabolic line ν . The homogeneous boundary conditions now imply that ϕ vanishes on all of $\partial\Omega^+$. The maximum principle for nonuniformly elliptic equations implies that ϕ vanishes identically on Ω^+ (see the Remark following the proof of Theorem 3.1 in Sec. 3.1 of [37]).

Denote by Γ the set of polar lines of γ , where γ is the set of chords $x = \tau$ for $x_0 < \tau < 1$. Then Γ is a set of characteristic lines to eq. (16), which are intersecting pairs of tangent lines to the unit circle. Order Γ by decreasing radial distance, from the origin, of the point of intersection for the two polar lines of τ . Then the indexed set Γ_λ , beginning with the polar lines of γ_0 and having a limit point at the vertical line $x = 1$, foliates the subdomain Ω^- in the sense that Ω^- can be expressed as the uncountable union of the elements of $\{\Gamma_\lambda\}$.

On elements of Γ_λ we have the characteristic equation

$$(21) \quad dr^2 - r^2 (r^2 - 1) d\theta^2 = 0,$$

so

$$d\chi = \chi_\theta d\theta + \chi_r dr = \left(\chi_\theta \pm \chi_r r \sqrt{r^2 - 1} \right) d\theta$$

and, using (17) and (18),

$$\begin{aligned} \frac{d\chi}{d\theta} &= r^2 (1 - r^2) \phi_r^2 - \phi_\theta^2 \mp 2r \sqrt{r^2 - 1} \phi_r \phi_\theta \\ &= - \left(r \sqrt{r^2 - 1} \phi_r \pm \phi_\theta \right)^2 \leq 0. \end{aligned}$$

Because $\chi_\theta = \chi_r = 0$ on ν , we know that χ is constant on ν . Integrating along elements of Γ_λ and using the constancy of χ on the arc ν now implies that χ is constant, and thus χ_θ vanishes, on all of Ω^- . But $\chi_\theta = \psi_1$ and $1 - r^2 < 0$ on Ω^- , so (17) implies that $\phi_r^2 = \phi_\theta^2 = 0$ on Ω^- . Thus ϕ is constant on Ω^- . Because ϕ vanishes on ν , we find by continuity that the constant is zero.

This completes the proof of Theorem 1.

Corollary 2. *Considering (16) as an equation on a domain Ω of the Cartesian $r\theta$ -plane, the closed Dirichlet problem for classical solutions of eq. (16) is over-determined on Ω .*

Remarks. *i)* A nonexistence result of an entirely different kind, for solutions of a semilinear elliptic-hyperbolic equation, is presented in [68]. That result is based on an elliptic-hyperbolic extension of the Pohozaev identities of elliptic theory.

ii) Defining the first-order operator $N\phi = -2\phi_r$ and the 1-form

$$\psi = \psi_1 d\theta + \psi_2 dr,$$

eq. (19) can be interpreted formally as associating a conservation law to the second-order operator $L\phi$ defined by eq. (16) via the method of multipliers:

$$0 = \int \int N\phi \cdot L\phi dr d\theta = \int \int d\psi.$$

(Compare this equation with p. 262 of [69] and eqs. (5), (6) of [80]; see also the equation preceding eq. (7.1) of [29], taking $\partial U/\partial t = 0$ in that equation.) However, this interpretation is only applicable to a sufficiently small coordinate patch about an arc of the sonic curve. In general, χ is only incompletely specified by eq. (19); however, in the case considered in Theorem 1, regions on which ϕ must vanish are deduced from boundary conditions and the particular geometry of the domain.

iii) Despite the close similarity of Theorem 1 to Sec. 3 of [82], eq. (16) differs from the corresponding equation, (23), of [82] in its type-change function and lower-order terms. Moreover, because the hyperbolic boundary depends on the form of the characteristic equation, this difference in type-change function affects the geometry of the domain.

iv) The characteristic equation (21), in its Cartesian form

$$(22) \quad (a^2 - x^2) dy^2 + 2xy dx dy + (a^2 - y^2) dx^2 = 0$$

for a a constant, seems to have been introduced in 1854 by G. G. Stokes, in the same Cambridge Smith's Prize examination in which Stokes' Theorem first appeared (*c.f.* [114] and eq. 1 of [95]). Although no geometric context was suggested in the examination question, Stokes drew the attention of examinees to the possibility of singular solutions.

3.2. A strongly well-posed boundary-value problem. Solutions which exist in the closure of the graph of the differential operator are said to be *strong*; *c.f.* [33], p. 354. Precisely, a *strong solution* of a boundary-value problem for an operator equation $Lu = f$, with $f \in L^2$, is an element $u \in L^2$ for which there exists a sequence u^ν of continuously differentiable functions, satisfying the boundary conditions, for which

$$\lim_{\nu \rightarrow \infty} \|u^\nu - u\|_{L^2} = \lim_{\nu \rightarrow \infty} \|Lu^\nu - f\|_{L^2} = 0.$$

A consequence of this definition is the uniqueness of strong solutions.

A system of two differential equations on a domain of \mathbb{R}^2 is *symmetric positive* [33] if it can be written in the form of the matrix equation

$$(23) \quad Lw \equiv A^1 w_x + A^2 w_y + Bw = \varphi,$$

where $w = (w_1, w_2)$ is a vector, the matrices A^1 and A^2 are symmetric, and the symmetric part κ^* of the matrix $\kappa \equiv B - (1/2)(A_x^1 + A_y^2)$ is positive definite. Here and below, an asterisk denotes symmetrization: $\kappa^* = (\kappa + \kappa^T)/2$.

Considering a solution ω to (23) as a mapping from the domain Ω to a finite-dimensional vector space V , denote by ∂V the restriction of V to the boundary $\partial\Omega$ of Ω . We will use in a fundamental way a theorem by Friedrichs [33]; see also [63] and [107].

Friedrichs' Theorem on Symmetric Positive Systems. *Let Ω be a bounded domain of \mathbb{R}^2 having C^2 boundary $\partial\Omega$. Consider the matrix $\beta = n_1 A^1_{|\partial\Omega} + n_2 A^2_{|\partial\Omega}$, where $n = (n_1, n_2)$ is the outward-pointing normal vector to $\partial\Omega$. Suppose that the matrix β admits a decomposition $\beta = \beta_+ + \beta_-$, for which the sum of the null spaces for β_+ and β_- spans ∂V ; $\mathfrak{R}_+ \cap \mathfrak{R}_- = 0$, where \mathfrak{R}_\pm are the ranges of β_\pm ; the matrix $\mu = \beta_+ - \beta_-$ satisfies $\mu^* \geq 0$. Then the boundary-value problem given by*

$$(24) \quad \begin{aligned} Lw &= \varphi \text{ in } \Omega, \\ \beta_- w &= 0 \text{ on } \partial\Omega, \end{aligned}$$

where L is given by (23), has a strong solution whenever the system is symmetric positive and the components of φ are square-integrable.

A boundary-value problem which possesses a strong solution is said to be *strongly well-posed*. We show the existence of such a problem for a class of equations of Keldysh type. We initially consider equations having the form

$$(25) \quad [K(\eta)u_\eta]_\eta + u_\xi\xi + ku_\xi = f(\eta, \xi),$$

where $K'(\eta) > 0$, k is a nonzero constant, $K(\eta) < 0$ for $0 \leq \eta < \eta_{crit}$ and $K(\eta) > 0$ for $\eta_{crit} < \eta \leq R$. We do not expect classical solutions, so rather than study this equation directly, we consider the associated system

$$(26) \quad Lw = A^1 w_\eta + A^2 w_\xi + Bw = F,$$

where L is a first-order operator, $w = (w_1(\eta, \xi), w_2(\eta, \xi))$, $F = (f, 0)$,

$$(27) \quad A^1 = \begin{pmatrix} K(\eta) & 0 \\ 0 & -1 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and

$$(28) \quad B = \begin{pmatrix} K'(\eta) & k \\ 0 & 0 \end{pmatrix}.$$

We interpret η as a radial coordinate and ξ as an angular coordinate, so that the system (26)-(28) is defined on a closed disc of radius R . The system is equivalent to (25) if the components of w are C^2 and $w_1 = u_\eta$, $w_2 = u_\xi$.

Theorem 3. *Suppose that there is a positive constant ν_0 such that $K'(\eta) \geq \nu_0$. Let there be continuous functions $\sigma(\xi)$ and $\tau(\xi)$ such that the boundary condition*

$$(29) \quad \sigma(\xi)w_1 + \tau(\xi)w_2 = 0,$$

where the product $\sigma(\xi)\tau(\xi)$ is either strictly positive or strictly negative and has sign opposite to the sign of k . Then the boundary-value problem (26)-(29), with K and k as defined in eq. (25), possesses a strong solution on the closed disc $\{(\eta, \xi) \mid 0 \leq \eta \leq R\}$ provided $|K(0)|$ is sufficiently small.

Proof. Multiply the terms of eq. (26) by the matrix

$$E = \begin{pmatrix} a & -cK(\eta) \\ c & a \end{pmatrix},$$

where a and c are constants; the sign of c is chosen so that $\sigma\tau c < 0$ (so $ck > 0$); $a > 0$; and $|c|$ is large. The matrix E is nonsingular provided

$$\det E = a^2 + c^2 K(\eta) \neq 0.$$

Because $K(\eta)$ is continuous, increasing, and $K(\eta_{crit}) = 0$, the invertibility condition for E becomes

$$\min_{\eta \in [0, R]} |K(\eta)| < \frac{a^2}{c^2}.$$

This condition will be satisfied provided $|K(0)|$ is sufficiently small.

The symmetric part κ^* of the matrix

$$\kappa = EB - (1/2) [(EA^1)_\eta + (EA^2)_\xi]$$

has determinant

$$\Delta = \frac{ak}{2} \left[cK'(\eta) - \frac{ak}{2} \right] \geq \frac{ak}{2} \left[c\nu_0 - \frac{ak}{2} \right],$$

so the resulting system is symmetric positive provided $|c|$ is sufficiently large. The proof will be complete once we show that the boundary conditions are admissible. Proceeding as in [116], choose the outward-pointing normal vector to have the form $n = K^{-1}(\eta)d\eta$.

Then on the boundary $\eta = R$,

$$\beta = \begin{pmatrix} a & c \\ c & -aK^{-1}(R) \end{pmatrix}.$$

Choose

$$\beta_- = \frac{1}{\sigma^2 + \tau^2} \begin{pmatrix} \sigma\tau c + \sigma^2 a & \tau^2 c + \sigma\tau a \\ -\sigma\tau aK^{-1}(R) + \sigma^2 c & -\tau^2 aK^{-1}(R) + \sigma\tau c \end{pmatrix}.$$

Choose $\beta_+ = \beta - \beta_-$. Then $\beta_- w = 0$, as (29) implies that $w_2 = -(\sigma/\tau)w_1$ on the circle $\eta = R$. Moreover,

$$\mu = \frac{1}{\sigma^2 + \tau^2} \begin{pmatrix} (\tau^2 - \sigma^2)a - 2\sigma\tau c & (\sigma^2 - \tau^2)c - 2\sigma\tau a \\ (\tau^2 - \sigma^2)c + 2\sigma\tau aK^{-1}(R) & (\tau^2 - \sigma^2)aK^{-1}(R) - 2\sigma\tau c \end{pmatrix},$$

implying that

$$\mu^* = \frac{1}{\sigma^2 + \tau^2} \begin{pmatrix} (\tau^2 - \sigma^2)a - 2\sigma\tau c & \sigma\tau a(K^{-1}(R) - 1) \\ \sigma\tau a(K^{-1}(R) - 1) & (\tau^2 - \sigma^2)aK^{-1}(R) - 2\sigma\tau c \end{pmatrix}.$$

If $\sigma\tau < 0$, choose $c > 0$; if $\sigma\tau > 0$, choose $c < 0$. Then the matrix μ^* will be non-negative provided $|c|$ is sufficiently large.

Now

$$\mathfrak{R}_- = \frac{\sigma\omega_1 + \tau\omega_2}{\sigma^2 + \tau^2} \begin{pmatrix} \tau c + \sigma a \\ -\tau aK^{-1}(R) + \sigma c \end{pmatrix}$$

and

$$\mathfrak{R}_+ = \frac{\tau\omega_1 - \sigma\omega_2}{\sigma^2 + \tau^2} \begin{pmatrix} \tau a - \sigma c \\ \sigma aK^{-1}(R) + \tau c \end{pmatrix},$$

so $\mathfrak{R}_- \cap \mathfrak{R}_+ = 0$. Because conditions are given on the entire boundary of the disk, the null space of β_- alone spans the range of ∂V .

The invertibility of E completes the proof of Theorem 3.

Remarks. *i)* Equation (25) is a lower-order perturbation of an equation studied by Magnanini and Talenti in the context of singular optics [71]. Those authors considered L^2 data prescribed on the boundary of the unit disc. They were able to

construct an explicit, smooth solution in the interior of the disc having a point singularity at the origin. Moreover, they showed the problem to be weakly well-posed on the entire domain. Theorem 3 of this section replaces Theorem 3 of [95] which claims, on the basis of an erroneous proof [96], that a strong solution to a closed boundary-value problem exists for an arbitrarily small lower-order perturbation of the equation studied in [71]. In the present theorem the perturbation is lower-order but not arbitrarily small.

ii) A similar result has been proven by Torre [116] for the equation

$$(30) \quad \frac{1}{\eta} (\eta u_\eta)_\eta + \left(\frac{1}{\eta^2} - \tilde{\omega}^2 \right) u_{\xi\xi} = f(\eta, \xi),$$

where f is a sufficiently smooth function and $\tilde{\omega}$ is a constant. In [116], condition (29) is imposed on functions σ, τ such that the product $\sigma\tau$ does not vanish on the outer boundary of an annulus and σ, τ satisfy the Dirichlet conditions $\sigma = 1, \tau = 0$ on the inner boundary. Because eq. (30) is a helically reduced wave equation in $\mathbb{M}^{2,1}$, it provides a toy model for the reduction of the Einstein equations by a helical Killing field (*c.f.* Examples 1 and 4 of Sec. 6, below). In distinction to eq. (25), which is of Keldysh type, eq. (30) is of Tricomi type.

This section has been devoted to closed boundary-value problems for harmonic fields on extended \mathbb{P}^2 and similar equations of Keldysh type. The vast literature on open boundary-value problems for equations of the form (13) and its nonlinear relatives has been ignored. The literature on such problems published during the first half of the twentieth century is reviewed in [12]; for more recent literature, see [17]. See also [72]. All three references concentrate on the literature in mathematical fluid dynamics, but results also exist on boundary-value problems for nonlinear equations related to (13) that arise in connection with the isometric embedding of Riemannian manifolds into a higher-dimensional Euclidean space [67],[84], [127].

4. DUALITY

In this and the following section we return to the geometric variational problems introduced in Section 2. But we will eventually replace the linear Hodge theory of eqs. (2) by a nonlinear Hodge theory based on an elliptic-hyperbolic extension of the Yang-Mills equations.

4.1. Hodge duality of surfaces. It has been known for nearly a century that every solution of the Euclidean minimal surface equation

$$(31) \quad \nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0$$

over all of \mathbb{R}^2 is an affine linear function [11]. Geometrically, this means that a non-parametric minimal surface of \mathbb{R}^3 is a plane. This *Bernstein property* was extended to entire solutions of the Lorentzian maximal space-like hypersurface equation

$$(32) \quad \nabla \cdot \left(\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right) = 0$$

by Calabi [15]: a maximal space-like hypersurface of $\mathbb{M}^{2,1}$ is a space-like plane. Calabi's result was later extended to space-like hypersurfaces of n -dimensional Minkowski space by Cheng and Yau [20].

Yang has interpreted this kind of equivalence between equations (31) and (32) in the language of differential forms ([126], eqs. (2.23)–(2.29); see also [1]). In Yang's interpretation, Calabi's equivalence becomes duality under the Hodge isomorphism.

Yang's argument involves writing eq. (5) in the dual form [108]

$$(33) \quad \delta [\rho(Q)\omega] = d\omega = 0,$$

where, as in (2), $d : \Lambda^p \rightarrow \Lambda^{p+1}$ is the exterior derivative with formal adjoint $\delta : \Lambda^p \rightarrow \Lambda^{p-1}$;

$$Q(\omega) = |\omega|^2 = *(\omega \wedge *\omega);$$

$*$: $\Lambda^p \rightarrow \Lambda^{n-p}$ is the Hodge involution. If the surface is embedded in Euclidean 3-space, then we choose

$$(34) \quad \rho(Q) = \frac{1}{\sqrt{1+Q}}.$$

If the surface is embedded in Minkowski 3-space, then

$$(35) \quad \rho(Q) = \frac{1}{\sqrt{|1-Q|}}.$$

In the latter case it is necessary to distinguish space-like surfaces, for which $Q < 1$, from time-like surfaces, for which $Q > 1$. The surface $Q = 1$ is the light cone. Note that none of these surfaces can be found in $\mathbb{M}^{2,1}$ without solving the system (33), (35).

If ω is a p -form satisfying (33) over a region having trivial de Rham cohomology, then the $n - p - 1$ -form $*[\rho(Q)d\omega]$ is closed, and thus is the differential of an $n - p - 2$ -form σ :

$$d\sigma = *[\rho(Q)d\omega].$$

If $\rho(Q)$ is given by (34), then this implies that

$$0 = d^2\omega = d \left[(-1)^{p(n-p)+n-1} * \frac{d\sigma}{\sqrt{1-|d\sigma|^2}} \right] = \pm \delta [\tilde{\rho}(Q)d\sigma],$$

with $\tilde{\rho}(Q)$ given by (35); see eqs. (2.23-2.30) of [126] for details.

Yang obtains from this duality, not the Bernstein property, but rather a *Liouville property* ([126], eq. (2.31)): if either $d\omega$ or $d\sigma$ have finite energy on \mathbb{R}^n in the sense that

$$(36) \quad E = \int_{\mathbb{R}^n} \int_0^Q \rho(s) ds * 1 < \infty$$

for $\rho(Q)$ given by either (34) or (35), then

$$(37) \quad d\omega = 0 \text{ or } d\sigma = 0.$$

See Secs. 1-4 of [110] for more details of this and similar Liouville theorems. We return to this topic in Sec. 5.

4.2. Hodge duality of flow metrics. It is also possible to establish a kind of Hodge duality between extremal surfaces in $\mathbb{M}^{2,1}$ and a fully quasilinear model for compressible potential flow. If L is the operator of eq. (3), we define as in [12] the *flow metric* for the operator L to be the metric tensor g_{ij} having distance element

$$ds_L^2 \equiv \alpha(x, y) dy^2 - 2\beta(x, y) dx dy + \gamma(x, y) dx^2.$$

The metric g_{ij} is Riemannian for regions on which L is an elliptic operator and Lorentzian for regions on which L is a hyperbolic operator.

This idea does not use the linearity of L in any essential way, and its original motivation seems to have been the quasilinear equation for the steady flow of an ideal gas. Define the quantity

$$c^2 = 1 - \frac{\gamma - 1}{2} (u^2 + v^2),$$

where u and v are the components of the velocity of an ideal compressible fluid in the x and y directions, respectively; $\gamma > 1$ is the adiabatic constant of the medium. (The notation, which is historical, implies that the flow metric has traditionally not been applied beyond the cavitation velocity $u^2 + v^2 = 2/(\gamma - 1)$. The gas is assumed to be adiabatic and isentropic in order to avoid having to specify thermodynamic variables.)

In the irrotational case we can define a potential function $\psi(x, y)$ for which

$$d\psi = udx + vdy$$

and

$$*d\psi = udy - vdx,$$

where the centered asterisk again denotes Hodge involution. Then the flow metric for the continuity equation of compressible ideal flow in the plane,

$$(c^2 - u^2) u_x - 2uvu_y + (c^2 - v^2) v_y = 0,$$

has distance element

$$(38) \quad ds^2 = c^2 (dx^2 + dy^2) - (*d\psi)^2;$$

the flow metric of the classical minimal surface equation

$$(39) \quad (1 + v^2) u_x - 2uvu_y + (1 + u^2) v_y = 0$$

for the graph of a smooth function $z = \psi(x, y)$ has distance element

$$(40) \quad ds'^2 = dx^2 + dy^2 + d\psi^2.$$

Each of the two flow metrics (38) and (40) decomposes into a conformally Euclidean part and a non-Euclidean part. Moreover, if the classical minimal surface equation (39) for smooth surfaces embedded in \mathbb{R}^3 is replaced by the equation (5) for surfaces embedded in $\mathbb{M}^{2,1}$, then the non-Euclidean part of that flow metric is the Hodge dual of the non-Euclidean part of the flow metric for ideal compressible fluids: the flow metric for (5) has element of distance ds'' given by

$$ds''^2 = dx^2 + dy^2 - d\psi^2.$$

For more details on duality of flow metrics, see Sec. 2.1 of [91].

The approach to duality initiated by eq. (33) seems to have been introduced in [108], in the context of compressible flow on a compact Riemannian manifold; also compare [54] and [78] with [109].

5. GAUGE INVARIANCE

5.1. Twisted nonlinear Hodge equations. It is natural to ask why the Bernstein property for extremal surfaces in [11], [15], and [20] becomes a Liouville property for differential forms in eq. (37). The Bernstein property for solutions of the two equations (31) and (32) has an intuitive interpretation because these partial differential equations can be embedded in a geometric theory, the theory of extremal hypersurfaces. In order to obtain an intuitive interpretation of Yang's Liouville property for solutions of the two systems (33), (34) and (33), (35), it would be useful to embed those partial differential equations as well in an appropriate geometric theory. Yang observes [126] that we can do so if the order of the differential form ω is 1 or $n - 1$. We can also do so in the case studied by Yang, in which the order of the differential form is 2.

In that case eq. (33), with ρ given by (35), is the Born-Infeld equation for an electromagnetic field ω [14], [126]. There is a generalization of the Born-Infeld equations which illuminates their geometric properties and, in particular, gives a geometric interpretation of the Liouville property for those equations.

Proceeding as in [89] and [90], we denote by X a vector bundle having compact structure group $G \subset SO(m)$ and define X over a smooth, finite, oriented, n -dimensional Riemannian manifold M . Form an admissible class of connections by choosing a smooth base connection D in the space of connections compatible with G and considering the class of connections $D + A$, where A is a section of $ad X \otimes T^*M$ which lies in the largest Sobolev space for which the energy functional E given by (36) is finite (with the domain \mathbb{R}^n replaced by M); details are given in [119] for the case $\rho \equiv 1$. Denote by F_A the curvature 2-form associated to a connection 1-form A . Then $Q = |F_A|^2 = \langle F_A, F_A \rangle$ is an inner product on the fibers of the bundle $ad X \otimes \Lambda^2(T^*M)$. The inner product on $ad X$ is induced by the normalized trace inner product on $SO(m)$ and the inner product on $\Lambda^2(T^*M)$ is induced by the exterior product $*(F_A \wedge *F_A)$. Sections of the automorphism bundle $Aut X$ are *gauge transformations*. These maps act tensorially on F_A but affinely on A ; see, *e.g.*, Sec. 6.4 of [73]. Note that it is possible to describe the geometry of F_A either in terms of a principal bundle or in terms of the associated vector bundle. The choice is a matter of convenience; see, *e.g.*, Sec. 2.3 of [73] for a discussion.

In the smooth case we take variations by computing $(d/dt)(F_{D+tA})$ at the origin of t . Using the fact that for any smooth section σ we have ([89], Sec. 1)

$$F_{D+tA}(\sigma) = (F + tDA + t^2 A \wedge A)(\sigma),$$

we obtain

$$\begin{aligned} \delta E &= \frac{1}{2} \int_M \rho(Q) \delta Q \, dM \\ (41) \quad &= \frac{1}{2} \int_M \rho(Q) \frac{d}{dt} \Big|_{t=0} |F + tDA + t^2 A \wedge A|^2 \, dM. \end{aligned}$$

Letting $t = 0$, the right-hand side of (41) can be written

$$(42) \quad \int_M \rho(Q) \langle DA, F_A \rangle \, dM = \int_M \langle DA, \rho(Q) F_A \rangle \, dM.$$

We assume that either $\partial M = 0$ or, if not, that F_A satisfies a “Neumann” boundary condition of the form

$$(43) \quad i^*(\ast F) = 0$$

on ∂M , where i^* is the pull-back under inclusion of the boundary of M in M . This is equivalent in local coordinates to prescribing zero boundary data for F in a direction normal to ∂M ; see [74] for details in the case $\rho \equiv 1$.

Set δE equal to zero. Then (41) and (42) imply

$$0 = \int_M \langle DA, \rho(Q)F_A \rangle dM = \int_{\partial M} A_{\vartheta} \wedge (\rho(Q)F_A)_N + \int_M \langle A, D^*(\rho(Q)F_A) \rangle dM,$$

where D^* denotes the formal adjoint of the exterior covariant derivative D ; ϑ denotes tangential component on the boundary and N , the normal component there. Condition (43) implies Euler-Lagrange equations of the form [87], [88]

$$(44) \quad D^*(\rho(Q)F) = 0.$$

Because F is a curvature 2-form, it satisfies an additional condition

$$(45) \quad DF = 0,$$

the second Bianchi identity.

If $\rho \equiv 1$, then eqs. (44), (45) degenerate to the Euclidean Yang-Mills equations of high-energy physics. Viewed in another way, one obtains eqs. (44), (45) from eqs. (33) by twisting the cotangent bundle in which solutions of (33) live. This twist is represented analytically by a nonvanishing Lie bracket with A . Precisely, $A \in \Gamma(M, ad X \otimes T^*M)$ is a connection 1-form on X having curvature 2-form

$$F_A = dA + \frac{1}{2} [A, A] = dA + A \wedge A,$$

where $[,]$ is the bracket of the Lie algebra \mathfrak{S} , the fiber of the adjoint bundle $ad X$. Thus eqs. (44), (45) can be characterized algebraically as a non-commutative version of (33), or geometrically as *twisted nonlinear Hodge equations*.

Equations (44), (45) change from elliptic to hyperbolic type along the curve

$$\frac{d}{dQ} (Q\rho^2(Q)) = 0.$$

So, in this case as well, the elliptic and hyperbolic regions of the equations cannot be found without solving the system. For ρ given by eq. (34), eqs. (44), (45) do not change type, but ellipticity degenerates as Q tends to infinity.

Because eqs. (33) are equivalent to eqs. (44), (45) in the special case of an abelian gauge group G , the closed 2-form ω of the classical Born-Infeld theory is, geometrically, an abelian special case of the curvature 2-form F_A (c.f. [90], Sec. 1). We can interpret Yang’s Liouville theorem given by eq. (37) for solutions of the system (33) with (34) or (35), in the generalized perspective of [88]-[90], as the assertion that a finite-energy solution of (33) on \mathbb{R}^n with density (34) or (35) is associated with a bundle having zero curvature, just as entire solutions in \mathbb{R}^3 or $\mathbb{M}^{2,1}$ are associated with surfaces having zero curvature. That geometric interpretation is reflected in the degeneration of the Bernstein property of [15], for entire solutions of (31) and (32), to a Liouville property in [126] for entire solutions of (33) with (34) or (35).

Liouville theorems have also been derived for solutions of (44) and similar geometric objects; see, *e.g.*, [86], [88], Sec. 5 of [93], and Secs. 3 and 4 of [110]. An n -dimensional Born-Infeld system related to time-like extremal surfaces in higher-dimensional Minkowski space is studied in [59].

In [56] a physical model for compressible ideal flow with $SO(3)$ gauge invariance is proposed. The mathematical structure of this model is similar in some ways to the variational theory presented here.

Choosing the function $\rho(Q)$ in eqs. (44) to be $1 + 4\gamma Q^{-1}$, where γ is a parameter, the gauge group to be the abelian group $U(1)$, and the underlying metric to be Robertson-Walker, eqs. (44) become identical to eqs. (12) of [85], a nonlinear electrodynamic model for the accelerated expansion of the universe.

A version of eqs. (44), (45) for mappings exists [90], [94], [97] which is related to harmonic maps in exactly the same way that eqs. (44), (45) are related to Yang-Mills fields. This class of maps is a special case of the F -harmonic maps introduced by Ara [5].

5.2. Global triviality on \mathbb{R}^n . In this section we extend the Liouville theorems cited in the preceding section to the case of an energy functional for which the variational equations may be elliptic-hyperbolic and the density may cavitate, in a pointwise sense, on a set of measure zero. The bundle connection need not be a pointwise solution of any equation at all in order to satisfy the hypotheses of this section.

Consider a principal bundle Π over a domain Ω of \mathbb{R}^n and a 1-parameter family ϕ^t of compactly supported diffeomorphisms of Ω for which $\phi^s \circ \phi^t = \phi^{s+t}$ with ϕ^0 the identity transformation. This family of reparametrizations can be lifted to the principal bundle by parallel transport, with respect to an arbitrary smooth connection, along the curve $x_s = \phi^s(x)$ from $x_0 = x$ to $x_t = \phi^t(x)$. If $A \in \Gamma(\Omega, \Lambda^1(ad \Pi))$ is a Lie-algebra-valued connection 1-form, then we define $A^t = (\psi^t)^* A$, where ψ^* is a lifting of ϕ^* to $\Gamma(\Omega, \Lambda^1(ad \Pi))$. (The superscripted asterisk denotes an induced mapping.) For details of this lifting, see [106]. We say that the connection A is r -stationary with respect to the energy functional E given by (36) if

$$\begin{aligned} \delta_r E &= \frac{d}{dt} \Big|_{t=0} E(F^t) = \frac{d}{dt} \Big|_{t=0} E(F(A^t)) \\ &= \frac{d}{dt} \Big|_{t=0} \int_{\Omega} \int_0^{|F^t|^2} \rho(s) ds * 1 = 0, \end{aligned}$$

c.f. [2]. The following theorem provides sufficient conditions under which A must have zero curvature almost everywhere if $\Omega = \mathbb{R}^n$. A slightly more complicated argument will extend the result to manifolds of constant negative curvature (*c.f.* [106]) and to $\mathbb{R}^n \setminus \Sigma$, where Σ is a singular set of prescribed codimension (*c.f.* [86]).

Theorem 4. *Consider the geometric construction of the preceding paragraph. Assume that ρ is continuously differentiable, nonnegative, and weightless under conformal transformations; that there is a positive number Q_{crit} such that*

$$(46) \quad e(Q) = \int_0^Q \rho(s) ds > 0 \quad \forall Q \in (0, Q_{crit}),$$

with $Q(F) = |F_A|^2$ defined as in Sec. 5.1; that the restriction of E to a Euclidean n -disc of radius R is r -stationary $\forall R > 0$; that $\rho'(s) \leq 0 \forall s \geq 0$; and that

$$(47) \quad E|_{B_R} \leq CR^k$$

as R tends to infinity for a positive constant C and a sufficiently small (positive) constant k . Then if Q is bounded above by Q_{crit} , $Q(F(x))$ must be zero for almost every $x \in \mathbb{R}^n$ for $n > 4$.

Proof. Similar theorems under somewhat different hypotheses have been proven in [106]; see also Secs. 2 and 5 of [88], and Sec. 5 of [93]. Assume that the lifting described earlier has been constructed. Denote by ψ^t the lift of a 1-parameter family of compactly supported diffeomorphisms of Ω such that

$$\psi^s \circ \psi^t = \psi^{s+t},$$

and $\psi^0 = \text{identity}$. Define

$$(48) \quad f \equiv \psi^t(x) = x + t\xi(x) + O(t^2),$$

where

$$\xi(x) = \frac{d}{dt}\bigg|_{t=0} \psi^t(x)$$

is the variation vector field, to be chosen. We have

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=0} F_{ij}(f) df^i df^j &= \frac{d}{dt}\bigg|_{t=0} F_{ij}(f) \frac{\partial f^i}{\partial x^k} dx^k \frac{\partial f^j}{\partial x^m} dx^m = \\ \frac{d}{dt}\bigg|_{t=0} F_{ij}(f) \left(\delta_k^i \delta_m^j + \delta_k^i t \frac{\partial \xi^j}{\partial x^m} + \delta_m^j t \frac{\partial \xi^i}{\partial x^k} + O(t^2) \right) dx^k dx^m. \end{aligned}$$

Make the coordinate transformation $x \rightarrow y$, where

$$y = (\psi^t)^{-1}(x).$$

Then

$$\frac{d}{dt}\bigg|_{t=0} F_{ij}(f) df^i df^j = 2F_{ij}(x) \frac{\partial \xi^i}{\partial x^k} dx^k dx^j.$$

If J is the Jacobian of the transformation $x \rightarrow y$, then

$$(49) \quad \frac{d}{dt}\bigg|_{t=0} J[(\psi^t)^{-1}] = \frac{d}{dt}\bigg|_{t=0} \left| \frac{\partial x}{\partial f} \right| = -\text{div } \xi.$$

By hypothesis, $\forall R > 1$,

$$\begin{aligned} 0 &= \delta_r E = \frac{d}{dt}\bigg|_{t=0} \int_{B_R} e(\langle F^t, F^t \rangle) J[(\psi^t)^{-1}] * 1 = \\ &\quad \int_{B_R} e(Q) \frac{d}{dt}\bigg|_{t=0} J[(\psi^t)^{-1}] * 1 + \\ (50) \quad &\quad \int_{B_R} e'(Q) 2 \left\langle \frac{d}{dt}\bigg|_{t=0} F_{ij}(f) df^i df^j, F_{\ell m}(f) df^\ell df^m \right\rangle * 1. \end{aligned}$$

Substitution of (49) into (50) yields

$$\int_{B_R} e(Q) \text{div } \xi * 1 = 4 \int_{B_R} e'(Q) \left\langle F_{ij}(x) \frac{\partial \xi^i}{\partial x^\ell} dx^\ell dx^j, F_{ij}(f) df^i df^j \right\rangle * 1.$$

Choose an orthonormal basis

$$\{u_i\}_{i=1}^n = \left\{ \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta_2}, \dots, \frac{\partial}{\partial \theta_n} \right\}$$

and let [106] $\xi = \eta(r)r \cdot \partial/\partial r$, where: r is the radial coordinate in a curvilinear system; $\eta(r) \in C_0^\infty[0, 1]$; $\eta'(r) \leq 0$; $\eta(r) = v(r/\tau) = 1$ for $r \leq \tau$, where τ is a number in the interval $(0, 1)$; there is a positive number δ for which $\eta(r) = 0$ whenever r exceeds $\tau + \delta$. For this choice of ξ ,

$$(51) \quad \int_{B_R} e(Q) (n\eta + r\eta') * 1 = 4 \int_{B_R} Q\rho(Q)\eta * 1 + 4 \int_{B_R} \rho(Q)r\eta' \left| \frac{\partial}{\partial r} \lrcorner F \right|^2 * 1.$$

Our hypothesis on the sign of ρ' implies that

$$(52) \quad \begin{aligned} Q\rho(Q) &= \int_0^Q \frac{d}{ds} (s\rho(s)) ds \\ &= \int_0^Q [s\rho'(s) + \rho(s)] ds \leq \int_0^Q \rho(s) ds = e(Q). \end{aligned}$$

Substitution of (52) into (51) yields

$$\int_{B_R} e(Q) (n\eta - 4\eta + r\eta') * 1 \leq 4 \int_{B_R} \rho(Q)r\eta' \left| \frac{\partial}{\partial r} \lrcorner F \right|^2 * 1.$$

By construction

$$r\eta'(r) = -\tau \frac{\partial}{\partial \tau} v\left(\frac{r}{\tau}\right) \leq 0.$$

This yields

$$0 \leq 4 \int_{B_R} \rho(Q)\tau \frac{\partial}{\partial \tau} v\left(\frac{r}{\tau}\right) \left| \frac{\partial}{\partial r} \lrcorner F \right|^2 * 1 \leq \int_{B_R} e(Q) \left[(4-n)\eta + \tau \frac{\partial}{\partial \tau} v\left(\frac{r}{\tau}\right) \right] * 1.$$

As δ tends to zero we obtain

$$0 \leq \left(4 - n + \tau \frac{\partial}{\partial \tau} \right) \int_{B_\tau} e(Q) * 1.$$

Multiply this last inequality by the integrating factor $\tau^{4-(n+1)}$ and integrate over τ between r_1 and r_2 . We find that

$$(53) \quad r_1^{4-n} E|_{B_{r_1}} \leq r_2^{4-n} E|_{B_{r_2}}.$$

We can write the growth condition (47) in the form

$$(54) \quad r^{4-n} E|_{B_r} \leq C r^{4+k-n},$$

where $4+k-n < 0$ for sufficiently small k . The right-hand side of (54) tends to zero as r tends to infinity. The left-hand side is nonnegative by construction. Thus the conformal energy $r^{4-n} E|_{B_r}$ tends to zero on \mathbb{R}^n . Because by (53) the conformal energy is nondecreasing for increasing r , we conclude that E is identically zero on \mathbb{R}^n . The vanishing of the energy on a ball of infinite radius implies the pointwise vanishing of Q almost everywhere by inequality (46). This completes the proof.

An exactly analogous result holds in the case of a velocity field for a steady polytropic flow of an ideal compressible fluid; see Sec. 5 of [93] for details.

5.3. Local existence and regularity. In this section we briefly review the fundamental analysis of eqs. (44), (45). For details, see [89]; see also Sec. 2 of [93].

The local existence of solutions has been shown in the case $\rho(Q) = Q^{(p-2)/2}$ for p exceeding $n/2$, where n is the dimension of the domain. In this case the energy functional is Palais-Smale, and the existence of weak solutions follows by conventional variational theory. The equations are nonuniformly elliptic for this choice of ρ . It is possible to show that, in general, any weak solution is Hölder continuous on compact subdomains whenever $F_A \in L^p$ for $p > n/2$ and ρ is chosen in such a way that the system is *uniformly* elliptic.

In the special case of an abelian gauge group, a *weak solution* of (44), (45) is any curvature 2-form F_A for which $\rho(Q)F_A$ is orthogonal in L^2 to the space of d -closed 2-forms $d\zeta \in L^2(B)$ such that $\zeta \in \Lambda^1$ has vanishing tangential data on ∂B . For a nonabelian gauge group, an obvious extension to inhomogeneous equations allows a weak solution to be defined by the equation

$$(55) \quad \int_B \langle d\zeta, \rho(Q)F_A \rangle * 1 = - \int_B \langle \zeta, * [A, * \rho(Q)F_A] \rangle * 1.$$

Here B is a Euclidean n -disc, so the natural hypothesis on the domain is that it can be covered by n -discs; for example, assume that the domain boundary has no cusps.

We briefly outline the proof of regularity: The first step is to derive a weak subelliptic estimate for $|F|^2$, using difference quotients to establish the existence of an $H^{1,2}$ -derivative. The subelliptic estimate allows one to conclude from a limiting argument that $|F|$ is locally bounded by Morrey's Theorem. Now we apply a mean-value inequality for Lie-algebra-valued sections, which is valid in the uniformly elliptic case of the equations (44), (45). The mean-value inequality is applied to points in the solution space, so that the L^2 -difference of weak solutions can be compared despite the nonlinearity of ρ . Using this inequality, it is possible to estimate the L^2 -difference between a 2-form weakly satisfying eqs. (33) and a bounded, weak solution of eqs. (44), (45). The latter is considered in an exponential gauge, fixed in a Euclidean n -disc B centered at the origin of coordinates in \mathbb{R}^n . Solutions to eqs. (33) have known regularity, and in an exponential gauge, $A(0) = 0$ and $\forall x \in B$

$$|A(x)| \leq \frac{1}{2} |x| \cdot \sup_{|y| \leq |x|} |F(y)|.$$

Thus elliptic estimates, and the minimizing property of the variance by the mean with respect to location parameters, eventually show that the L^2 -difference of F and its local mean value decays sufficiently rapidly as the radius of the underlying domain shrinks to zero that Hölder continuity can be obtained from the Campanato Theorem. The final step is to show that the Campanato estimate is preserved under continuous gauge transformations in a small n -disc centered at a point close to the origin. This allows one to apply a covering argument which will extend to the entire domain the estimate that was obtained in an exponential gauge at the origin.

5.4. Other symmetry groups. Although the system (44), (45) is invariant under the action of the gauge group G and has the potential to change from elliptic to hyperbolic type, the system is so poorly understood that it is hard to know how these two properties affect solutions. In principle, the system cannot even be said to be elliptic-hyperbolic, as the type of the equation can only be fixed modulo

gauge transformations. However, the gauge invariance can be broken and (44), (45) represented as an elliptic-hyperbolic system whenever the curvature F_A lies in a sufficiently high Sobolev space. It is this property, which is shared with the Yang-Mills equations [119], that allows one to formulate sufficient conditions for the existence of solutions, and their regularity in the uniformly “subsonic” case, as described in the preceding section.

The distance element on the Beltrami disc is also invariant under the action of a gauge group, the group of projective transformations. This invariance has the consequence that the parabolic line for the Hodge equations (9), (10) on extended \mathbb{P}^2 is gauge-invariant under perspective to any conic section. So (9), (10) is an example of an elliptic-hyperbolic system in which gauge invariance has a measurable effect on the geometry of the problem. One might try to interpret a well known model for wave propagation in cold plasma, in which the parabolic line of an elliptic-hyperbolic system is a parabola with vertex at the origin of \mathbb{R}^2 [82], [105], as a fixed gauge of a Hodge system on extended \mathbb{P}^2 .

Unfortunately, there would be some problems with interpreting the cold plasma model in terms of waves on extended \mathbb{P}^2 . First, it does not appear to simplify the original problem, although it may illuminate certain similarities in the analysis of the two systems; compare [91], [92], and [98]. Second, the projective group consists of non-Euclidean motions, which are noninertial and thus lacking in obvious physical meaning, so it is unclear how to interpret the mathematical relation between the two systems in terms of a satisfying physical theory. Finally, the gauge groups of Sec. 5.1 act “upstairs” on a fiber bundle of states. The symmetry group under which the Laplace-Beltrami equations are invariant acts “downstairs” on the underlying metric, in the manner of the gauge group of general relativity. Thus it is reasonable to look for physical analogies of Sec. 2 among theories in which the relevant bundle is soldered to the base space rather than being related to the base space only by a projection, as in plasma physics, the Born-Infeld model, and other electromagnetic theories.

Symmetry groups for differential operators on Riemannian-Lorentzian metrics other than extended \mathbb{P}^2 have been computed [69]. In particular, the symmetry group for operators of Tricomi type which satisfy a pure power law has been shown to correspond to a group of local conformal transformations with respect to the underlying metric away from the metric singularity; moreover, the group extends across the singular surface on which the metric changes type [102].

6. A ZOO OF MIXED RIEMANNIAN-LORENTZIAN METRICS

For the reasons cited in the preceding section, it is reasonable to look to relativity for physical analogies of Sec. 2. We therefore include a brief survey of signature change in the recent physics literature.

1. *Special relativity*: The wave equation on Minkowski space-time, in a reference frame rotating with constant angular velocity ω with respect to another reference frame, is expressible in cylindrical coordinates (ρ, φ, z) as the elliptic-hyperbolic equation [111]

$$\frac{1}{\rho} (\rho u_\rho)_\rho + \left(\frac{1}{\rho^2} - \omega^2 \right) u_{\varphi\varphi} + u_{zz} = 0.$$

2. *Quantum cosmology*: These examples arise from the (controversial) *Hartle-Hawking hypothesis* [46], that the universe might have originated as a manifold having Euclidean signature and subsequently undergone a transition to a model having Lorentzian signature across a hypersurface which was space-like as seen from the Lorentzian side. (Note that certain metrics which are called *Euclidean* by physicists would be called *Riemannian* by geometers; see footnote 2 of [36].) Some 2-dimensional variants are [111]:

i) continuous change of signature:

$$ds^2 = -tdt^2 + dz^2;$$

ii) discontinuous change of signature:

$$ds^2 = -z^{-1}dt^2 + dz^2;$$

iii) continuous change of signature with a curvature singularity:

$$ds^2 = -zdt^2 + dz^2;$$

see also [27].

These examples have obvious higher-dimensional analogues.

Alternatively, the metric might change from Lorentzian to Kleinian signature across the line $z = 0$:

$$ds^2 = -dt^2 + dx^2 + dy^2 + z dz^2;$$

this 4-dimensional model has been studied in, *e.g.*, [3].

In connection with the distinction between examples *ii*) and *iii*), we note that operators on Riemannian-Lorentzian metrics which degenerate rather than blow up at the change of signature have been studied by mathematicians as well as physicists [19]; see also [18] and the discussion in Sec. 5 of [102].

It is interesting for mathematicians that although many of the criticisms of Hartle and Hawking's widely discussed proposal concern physical predictions that it implies, controversy also arises from mathematical ambiguities in geometric analysis on mixed Riemannian-Lorentzian metrics and from the variety of potential singularities of such metrics. See [99] and the references therein for a recent discussion of the physical predictions; see, *e.g.*, [26], [27], [31], [32], [47], [50], [51], and [62] for discussions of the mathematical ambiguities. It has been observed [30] that singularities similar to those that are associated with the Hartle-Hawking transition can also arise in classical relativity, as in Example 1, above; see also [24], [25], [28], [48], [60], and [61].

3. *Repulsive singularities in 4-dimensional extended supergravity* [35]: Mathematically, this model is essentially a combination of 2*ii*) and 2*iii*);

4. *Binary black hole space-times with a helical killing vector* [58]: This model is a generalization of example 1; see also [52] and [116].

5. *Brane worlds*: A *brane* is a submanifold of a *bulk*, or higher-dimensional space-time. Traditionally, branes have been represented as uniformly time-like, but it has been observed [75] that they need not be time-like everywhere and provide a natural context for signature change. In fact, mixed Euclidean-Lorentzian branes can be constructed in such a way that both the bulk and the brane are regular. However, if viewed from within the brane, the change of signature may appear as a curvature singularity [36], [76]. For certain choices, this provides an elegant kinematic model

for both the apparent big bang singularity and the apparent accelerated expansion of the universe; in particular, no hypothesis of dark energy is needed to account for accelerated expansion [77].

Signature change has also been investigated in the context of spinor cosmology; see [120] for a recent example.

There is a substantial literature on constructing analogies for the dynamical equations for light in curved space-time using equations from models of condensed matter, including (but by no means limited to) Bose-Einstein condensates with a sink or a vortex [7], [66], [121]-[125]. These lead to analogies between acoustic waves in matter and wave equations on mixed Riemannian-Lorentzian manifolds which are reminiscent of those between gas dynamics and extremal surfaces discussed in Sec. 4.2.

The idea behind these models is elegantly simple: An acoustic wave has a relation to the flow in which it propagates which is analogous in some crucial ways to the relation between a light wave and the ambient space-time. This analogy yields a kinematic model for certain relativistic effects. For example, if the flow becomes supersonic, an acoustic wave emitted downstream from a listener will be trapped in an analogous way to the trapping of light inside a black hole with respect to an external observer.

Such analogies can be traced back at least to the flow metrics of [12] and possibly to the electrodynamics of [38]; a review is given in [6].

An elliptic-hyperbolic system associated with the Einstein evolution equations is studied in [4]. But in that example an elliptic gauge-fixing condition is coupled to hyperbolic evolution equations on a Lorentzian metric. Because the signature of the metric is fixed, the resulting system is qualitatively different from the cases considered here.

An important consideration for physical applications is whether the elliptic-hyperbolic differential operator is of *real principal type*, in which case the principal symbol of the operator is real-valued and no complete null bicharacteristic can be trapped over any compact subset of the domain. Because a null bicharacteristic is an integral curve of a Hamiltonian system canonically associated to the principal symbol, the major analytic properties for operators of real principal type depend only on the principal symbol, and not on the form of the lower-order terms; see the concluding remarks in Sec. 3 of [101].

If the operator is of real principal type, ideas from microlocal analysis can be applied to construct a natural theory of boundary regularity that is applicable to the elliptic-hyperbolic case [100]; see also [39] and [40]. A major difference between most of the physical examples listed here and the geometric examples of Secs. 1-5 is that the latter operators are not of real principal type; as a result, microlocal arguments appear to fail and what can or cannot be said about solutions tends to depend delicately on the precise form of the lower-order terms. A typical example is Theorem 3 of [41]. In particular, this kind of dependence prevents the derivation of uniqueness theorems by the expected arguments; see Sec. 5.1 of [98] for an illustration.

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